Double heavy-quarkonium production from electron-positron annihilation in the Bethe-Salpeter formalism

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In this work we evaluate the cross section of the process $e^+e^- \to J/\psi\,\eta_c$ at energy $\sqrt{s} \approx 10.6$ GeV in the Bethe-Salpeter formalism. To simplify our calculation, the heavy quark limit is employed. Without taking the beyond-leading-order contribution(s) into account, the cross section calculated in this scenario is comparable with the experimental data. We also present our prediction for the cross section of double bottomonium production $e^+e^- \to \Upsilon(1S)\eta_b$ for the energy range of $\sqrt{s} \approx (25\text{-}30)$ GeV which may be experimentally tested, even though there is no facility of this range available at present yet.

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I. INTRODUCTION

It is well known that there is a significant discrepancy between the experimental measurements [1, 2] and the NRQCD predictions [3, 4] for the process $e^+e^- \to J/\psi\,\eta_c$ at energy $\sqrt{s} \approx 10.6$ GeV. To reduce the discrepancy many efforts have been made. For example, as discussed in Ref. [5], corrections from pure electromagnetic interactions are introduced into the non-relativistic QCD (NRQCD) factorization formalism; the next-to-leading-order contributions of strong interaction are taken into account in Refs. [6, 7]; in Ref. [8], the authors take into account corrections to the J/ψ leptonic decay width and the scale dependence of the leading-order prediction; etc.

Indeed, NRQCD should work well while dealing with the processes where heavy flavors are involved, especially in this concerned reaction where only heavy flavors exist. In the scenario of NRQCD an expansion is carried out with respect to powers of the relative three-velocity v which reflects the relativistic effects. Even though, in this reaction, the relativistic effects obviously are smaller than in the processes dominated by light flavors, they still may cause sizable contributions. Later works confirm this viewpoint. One needs, therefore, to incorporate the relativistic corrections into the theoretical calculations in an appropriate way. There are various ways to take into account the relativistic corrections. Ebert et al. carefully studied the corrections from relativistic effects [9] and indicated that with the relativistic effects being properly handled, a prediction for the cross section which is consistent with the present experimental measurement can be obtained.

Moreover, the factorization scheme should work in this case, because the production of $J/\psi + \eta_c$ from e^+e^- via electromagnetic interaction can be regarded as a reversed Drell-Yan process where the factorization is proved. Thus one can factorize the hard and soft processes and then a convolution integration over the two parts results in the final amplitude. Therefore, one only needs to consider the relativistic corrections appearing in the soft part, i.e. in the hadronization process because the hard part is carried out in the framework of quantum field theory which is completely in the relativistic covariant form.

To understand the experimental results, many authors have proposed various projects to improve the theoretical framework in addition to NRQCD. For example, this problem was discussed with the method of perturbative QCD (pQCD) . In Ref. [10], the corrections of higher-twist wave functions were included in pQCD and the light-front quark model.

Discussions given in Refs. [6, 7, 11, 12] suggest that to reduce the large discrepancy between the experimental results and the theoretical predictions based on NRQCD for the exclusive process with the final state which is composed of two charmonia, large next-to-leading-order (NLO) corrections may appear (the 'NLO' contribution is about 1.8 to 2.1 times of the leading-order one). Including this large NLO contributions, their results are close to the lower bound set by the Babar and Belle collaborations for the double-charmonia production. The authors also indicate that including the relativistic corrections can further enhance the estimated value.

We would rather incline to believe that because of the validity of the factorization, perturbative calculation is suitable and, therefore, in our calculation we will ignore the contribution of the next-to-leading order corrections. Then one should expect that a larger correction may come from the soft part, i.e. the relativistic effect may be significant.

As mentioned before, Ebert et al. considered such effects [9]. Alternatively, in this work, for properly incorporating relativistic effects, we try to evaluate this exclusive process in the Bethe-Salpeter (BS) formalism [13]. The BS equation is in principle established in the framework of relativistic quantum field theory, therefore, it is supposed to include all relativistic effects. Of course, to solve this equation in practice, one needs to adopt some approximations such as the instantaneous approximation

where part of the relativistic effects are lost. However, in many cases, such loss is not serious. Indeed, the BS formalism is suitable for studying the binding systems composed of two heavy charm and anti-charm quarks (or bottom flavors). The transition amplitudes can be obtained, in a natural way, to be an overlap integration of BS wave functions (see e.g. Ref. [14]). In order to simplify the calculation, we will further impose the heavy quark limit [15, 16, 17], i.e. all the $1/m_Q$ corrections are neglected in the calculation in this paper. Under this limit, our result shows that, at the leading order of α_s, the theoretical prediction is comparable with the experimental result [1, 2]. We attribute this mainly to the inclusion of relativistic effects in our formalism. We will come back to give more discussion on this point in the last section.

The remainder of this paper is organized as follows. In Sec. II, we will study the BS equations for the vector and pseudo-scalar quarkoniums. In Sec. III, we will calculate the cross section of the process $e^+e^- \to J/\psi\,\eta_c$ in the BS formalism. In this section, we will also give the prediction for the much smaller cross section of the exclusive process with double bottomonium production, $e^+e^- \to \Upsilon(1S)\eta_b$. Sec. IV is reserved for our conclusions and discussions.

II. BS EQUATIONS FOR HEAVY QUARKONIUMS

The BS wave function for a meson which is composed of a quark and an anti-quark is defined as

$$\chi_{_P}(x_1,x_2)_{\alpha\beta} = \frac{\delta_{ij}}{\sqrt{3}}\,\langle 0|\mathrm{T}\,\psi^i_\alpha(x_1)\overline{\psi}^j_\beta(x_2)|P\rangle\,. \tag{1}$$

where P is the momentum of the meson, $\psi^i_\alpha(x_1)$ and $\overline{\psi}^j_\beta(x_2)$ are the field operators of the quark and anti-quark, respectively, α,β are Dirac spinor indices, and i,j denote the color indices. In momentum space the BS equation for the wave function $\chi_{_P}(x_1,x_2)_{\alpha\beta}$ can be written as (see Refs. [18, 19] for example)

$$\begin{aligned}\chi_{_P}(p) &= \frac{i}{\not p_1 - m_1 + i\epsilon}\int\frac{d^4k}{(2\pi)^4}V_P(p,k)\chi_{_P}(k)\\ &\quad\times\frac{i}{\not p_2 - m_2 + i\epsilon}\,,\end{aligned}\tag{2}$$

where p,k represent the relative momenta between the quark and anti-quark, $m_1(p_1)$ and $m_2(p_2)$ are the masses (momenta) of the quark and anti-quark, respectively, and $V_P(p,k)$ is the kernel. The spinor indices are suppressed for simplicity. For the heavy quarkonium $Q\overline{Q}$ ($Q=c,b$) studied in this paper, we have $m_1=m_2=m_Q$ and then $p_1 = P/2+p$ and $p_2=-P/2+p$.

To simplify the calculation we will take the heavy quark limit throughout this paper. In this limit, the propagators of the heavy quark and heavy anti-quark ($S(p_1)$ and $S(p_2)$, respectively) can be simplified in the following way [20]:

$$\frac{1}{\not p_1 - m_Q + i\epsilon} \to \frac{(1+\not v)/2}{p_\ell + E_0/2 + i\epsilon}\,, \tag{3}$$

$$\frac{1}{\not p_2 - m_Q + i\epsilon} \to \frac{-(1-\not v)/2}{p_\ell - E_0/2 - i\epsilon}\,, \tag{4}$$

where we have defined the binding energy $E_0 = M - 2m_Q$, $v = P/M$ is the 'four-velocity' of the meson, M is the mass of the meson and $p_\ell = v\cdot p$. With this simplification, the BS equation (2) becomes

$$\begin{aligned}\chi_{_P}(p) &= \frac{(1+\not v)/2}{p_\ell + E_0/2 + i\epsilon}\int\frac{d^4k}{(2\pi)^4}V_P(p,k)\chi_{_P}(k)\\ &\quad\times\frac{(1-\not v)/2}{p_\ell - E_0/2 - i\epsilon}\,.\end{aligned}\tag{5}$$

From this expression, one can easily see that the BS wave function $\chi_{_P}$ satisfies $\not v\chi_{_P} = \chi_{_P}$ and $\chi_{_P}\not v = -\chi_{_P}$. Then, in the heavy quark limit, similar to the case for the di-quark system studied in Ref. [21], we have the following very simple parameterizations for the BS wave functions of vector and pseudoscalar $Q\overline{Q}$ mesons (which are denoted by the subscripts "a" and "b", respectively):

$$\chi^{(s)}_{_{P_a}}(p) = (1+\not v)\not\varepsilon^{(s)}M_a f_a(p)\,,\quad J^{PC}=1^{--}\,, \tag{6}$$

$$\chi_{_{P_b}}(p) = (1+\not v)\gamma_5 M_b f_b(p)\,,\quad J^{PC}=0^{-+}\,, \tag{7}$$

where $\varepsilon^{(s)}$ is the polarization vector of the vector quarkonium which is orthogonal to the velocity, $\varepsilon^{(s)}\cdot v = 0$, f_a and f_b are scalar functions of the relative momentum p.

In this work, following the standard procedure for solving the BS equation we impose the instantaneous approximation onto the kernel as $V_P(p,k) = V(\mathbf{p},\mathbf{k})$. Taking the concrete steps given in Ref. [18], we can obtain this kernel by a Fourier transformation of the Cornell potential which contains a linear piece and a Coulomb-type piece, $-iV(r) = I\otimes IV_s(r) + \gamma^\mu\otimes\gamma_\mu V_v(r)$, where $V_s(r) = \lambda r + V_0$ and $V_v(r) = -\frac{4}{3}\frac{\alpha_s}{r}$. To avoid any infrared divergence, a convergent factor $e^{-\beta r}$ is introduced into the potential,

$$V_s(r) = \frac{\lambda}{\beta}(1-e^{-\beta r}) + V_0\,,\quad V_v(r) = -\frac{4}{3}\frac{\alpha_s}{r}e^{-\beta r}. \tag{8}$$

After the Fourier transformation, the kernel in momentum space reads [18]: $-iV(\mathbf{q}) = I\otimes IV_s(\mathbf{q}) + \gamma^\mu\otimes\gamma_\mu V_v(\mathbf{q})$, where

$$\begin{aligned}V_s(\mathbf{q}) &= -(\lambda/\beta + V_0)\delta^3(\mathbf{q}) + \frac{\lambda}{\pi^2}\frac{1}{(\mathbf{q}^2+\beta^2)^2}\,,\\ V_v(\mathbf{q}) &= -\frac{2}{3\pi^2}\frac{\alpha_s(\mathbf{q})}{(\mathbf{q}^2+\beta^2)},\end{aligned}\tag{9}$$

and the effective coupling constant is given by $\alpha_s(\mathbf{q}) = 4\pi/9\log\left(a + \frac{\mathbf{q}^2}{\Lambda^2_{QCD}}\right)$ with a being a parameter which freezes the running coupling constant at low energy.

With this kernel, the BS equation (5) is written as

$$\begin{aligned}\chi_{_P}(p) \;=\; & \frac{i}{(p_{_\ell}+E_0/2+i\epsilon)(p_{_\ell}-E_0/2-i\epsilon)}\frac{1+\not{v}}{2} \\ & \times\int\frac{d^4k}{(2\pi)^4}\Big[V_s(\mathbf{q})\chi_{_P}(k)+V_v(\mathbf{q})\gamma^\mu\,\chi_{_P}(k)\,\gamma_\mu\Big] \\ & \times\frac{1-\not{v}}{2}\,, \qquad (10)\end{aligned}$$

where $\mathbf{q}=\mathbf{p}-\mathbf{k}$. Substituting the BS wave functions (6) and (7) into Eq. (10) we get the following component equations for vector and pseudo-scalar quarkonia:

$$\begin{aligned}f_{a(b)}(p) \;=\; & \frac{i}{(p_{_\ell}+E_0/2+i\epsilon)(p_{_\ell}-E_0/2-i\epsilon)} \\ & \times\int\frac{d^4k}{(2\pi)^4}(V_s-V_v)(\mathbf{q})\,f_{a(b)}(k)\,. \qquad (11)\end{aligned}$$

For later convenience, here we also write out the BS equation for the conjugate wave function,

$$\begin{aligned}\overline{\chi}_{_P}(p) \;=\; & \frac{i}{(p_{_\ell}+E_0/2+i\epsilon)(p_{_\ell}-E_0/2-i\epsilon)}\frac{1-\not{v}}{2} \\ & \times\int\frac{d^4k}{(2\pi)^4}\Big[V_s(\mathbf{q})\overline{\chi}_{_P}(k)+V_v(\mathbf{q})\gamma^\mu\,\overline{\chi}_{_P}(k)\,\gamma_\mu\Big] \\ & \times\frac{1+\not{v}}{2}\,. \qquad (12)\end{aligned}$$

To solve the BS equation (11), for convenience, we can choose a coordinate frame in which the binding system is static. The BS wave functions in this frame are given in Eqs. (6) (7), where $f_a(p)$ and $f_b(p)$ are Lorentz scalar functions. Then the longitudinal component $p_{_\ell}=p^0$. After carrying out the integrations over p^0 and k^0 on both sides of the BS equation along some proper contour, Eq. (11) becomes [1]

$$\widetilde{f}_{a(b)}(\mathbf{p})=\frac{1}{E_0}\int\frac{d^3\mathbf{k}}{(2\pi)^3}(V_v-V_s)(\mathbf{q})\,\widetilde{f}_{a(b)}(\mathbf{k})\,, \qquad (13)$$

where we have defined the instantaneous wave functions by $\widetilde{f}_{a(b)}(\mathbf{p})=\int\frac{dp^0}{2\pi}f_{a(b)}(p)$.

Normalization of the BS wave functions.

The normalization condition of the BS wave function $\chi_{_P}$ for a vector meson can be written as

$$\begin{aligned}& i\int\frac{d^4p\,d^4p'}{(2\pi)^8}\overline{\chi}^{(s)}_{P_a}(p)\left[\frac{\partial}{\partial P^0_a}I_{P_a}(p,p')\right]\chi^{(s')}_{P_a}(p') \\ & =2P^0_a\delta_{ss'}\,, \qquad (14)\end{aligned}$$

where $I_P(p,p')=-(2\pi)^4\delta^4(p-p')\,S^{-1}(p_1)S^{-1}(p_2)$, s and s' are the spin indices of the vector meson, and $P^0_a=\sqrt{\mathbf{P}^2_a+M^2_a}$ is the energy of the bound state. Multiplying by $\delta_{ss'}$ on both sides and summing over s and s', one has the following normalization equation (in the static frame of the meson):

$$\frac{8\,v^0_a}{E^2_0}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,(\widetilde{f}_{a1}-\widetilde{f}_{a2})^2=2P^0_a\,, \qquad (15)$$

for the vector meson. $\widetilde{f}_{a1}$ and $\widetilde{f}_{a2}$ are defined as follows,

$$\widetilde{f}_{a1}(\mathbf{p})=M_a\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,V_s(\mathbf{p}-\mathbf{k})\widetilde{f}_a(\mathbf{k})\,, \qquad (16)$$

$$\widetilde{f}_{a2}(\mathbf{p})=M_a\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,V_v(\mathbf{p}-\mathbf{k})\widetilde{f}_a(\mathbf{k})\,. \qquad (17)$$

From the BS equation (13), $\widetilde{f}_{a1}-\widetilde{f}_{a2}=-E_0M_a\widetilde{f}_a$, one can see that the normalization equation (15) can be written as

$$4\,M_a\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\widetilde{f}_a(\mathbf{p})^2=1\,. \qquad (18)$$

The normalization equation for the pseudo-scalar meson has completely the same form as Eq. (18) (with the subscript "a" replaced by "b").

III. DOUBLE QUARKONIUM PRODUCTION FROM e^+e^- ANNIHILATION

Now we turn to discuss the exclusive process in the electron-positron collisions with the final state of two heavy quarkonia. The relevant Feynman diagrams for the process $e^+e^-\to J/\psi\,\eta_c$ are depicted in Fig. 1. Since

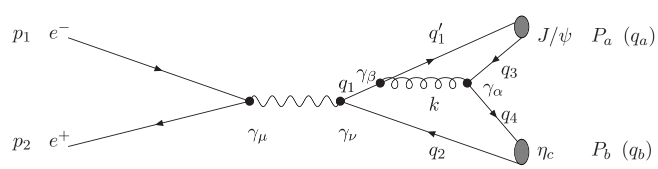

FIG. 1: Feynman diagrams for double charmonium production from electron-positron annihilation.

the s-channel gluon turns into $c\bar{c}$ in this process, hence the gluon is hard and the energy scale for this process is large. Therefore, the perturbative calculation in QCD to lowest order of α_s expansion is expected to be sufficient. We will not consider the contributions from diagrams of higher-order in α_s. Furthermore, we do not consider any higher order corrections from the pure electromagnetic interactions, which should be small and were discussed in Ref. [5].

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[1] In pratical calculation, to obtain the reasonable BS wave functions, we will replace E_0 by $M_{a(b)}-2\sqrt{\mathbf{k}^2+m^2_Q}$ in the following BS equation. This replacement is equivalent to regaining some part of $1/m_Q$ effects in the calculation.

One of the amplitudes in Fig.1 can be written as

$$\mathcal{M}_1 = C\,\frac{g^{\mu\nu} e\, e_Q g_s^2}{s}\,\bar v(p_2)\gamma_\mu u(p_1)\int\frac{d^4q_a d^4q_b}{(2\pi)^8} \times \mathrm{Tr}\left[\overline{\chi}_{P_a}\gamma_\beta\,\frac{1}{\not q_1 - m_c}\,\gamma_\nu\,\overline{\chi}_{P_b}\gamma_\alpha\right]\frac{g^{\alpha\beta}}{k^2}. \quad (19)$$

where $C = 4/3$ is the color factor, $\sqrt{s}$ is the total energy in the center-of-mass frame, and e_Q is the electric charge of the quark Q. The conjugation of the BS wave function is defined by $\overline{\chi}_P \equiv \gamma^0(\chi_P)^\dagger\gamma^0$. The momenta of the quark and anti-quark within the final state are $q_1' = \frac{1}{2}P_a + q_a$, $q_3 = \frac{1}{2}P_a - q_a$, $q_4 = \frac{1}{2}P_b + q_b$, $q_2 = \frac{1}{2}P_b - q_b$, and the momenta in the gluon and the quark propagators are given by

$$k = q_3 + q_4 = \frac{1}{2}(P_a + P_b) - q_a + q_b\,, \quad (20)$$

$$q_1 = q_1' + k = P_a + \frac{1}{2}P_b + q_b\,, \quad (21)$$

respectively. Since an integration is needed to obtain the amplitude (19) and since the propagators of the quark and the gluon depend on the relative momenta q_a and q_b one can expect that the calculation is very complicated. To simplify the calculation, we assume that the propagators of the quark and the gluon are independent of relative momenta q_a and q_b (see, e.g. Ref. [21]). This simplification is appropriate since the masses of heavy quarks are large compared with the relative momenta, which are of order $\alpha_s m_Q$. Then the momenta q_1 and k of the propagators are large compared with the relative momenta q_a and q_b. One may expect that, in the heavy quark limit, the calculation without taking into account the relative momenta should be exact. [2]

With the above approximation in mind, the momenta k and q_1 can be written as: $k \approx (P_a + P_b)/2$, $q_1 \approx P_a + P_b/2$ which lead to $k^2 \approx s/4$, $q_1^2 \approx s/2 + M_a^2/2 - M_b^2/4 \approx s/2 + m_c^2$. Furthermore, we will make use of the approximation $M_{J/\psi} \approx M_{\eta_c} \approx 2m_c$ in the calculation throughout this paper. Then the amplitude (19) can be written as

$$\mathcal{M}_1 = \frac{2^6}{3^2 s^3} g_s^2 e^2\,\bar v(p_2)\gamma_\mu u(p_1)\int\frac{d^4q_a}{(2\pi)^4}\frac{d^4q_b}{(2\pi)^4} \times \mathrm{Tr}\left[\overline{\chi}^{(s)}_{P_a}(q_a)\gamma_\alpha\,(\not q_1 + m_c)\gamma^\mu\overline{\chi}_{P_b}(q_b)\gamma^\alpha\right]. \quad (22)$$

From the component expressions of the BS wave functions (6) and (7), we can see $\overline{\chi}^{(s)}_{P_a} = \not\varepsilon^{(s)}(1 + \not v_a)M_a f_a$ and $\overline{\chi}_{P_b} = \gamma_5(1 + \not v_b)M_b f_b$. [3] Substituting the BS wave functions into the trace in the above amplitude, one achieves

$$\mathcal{M}_1 = -i\,\frac{2^{14}\pi^2}{3^2}\,\frac{\alpha_s\alpha_{\rm em}}{s^3}\,m_c\,\epsilon_{\mu\nu\rho\sigma}\,\varepsilon^{(s)\nu}P_a^\rho P_b^\sigma \times \bar v(p_2)\gamma_\mu u(p_1)\psi_a\psi_b\,, \quad (23)$$

where $\alpha_s = g_s^2/4\pi$, $\alpha_{\rm em} = e^2/4\pi$, ψ_a and ψ_b are two numbers defined by the integrations over f_a and f_b, respectively,

$$\psi_{a(b)} = \int\frac{d^4q}{(2\pi)^4}\,f_{a(b)}(q) = \int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\widetilde{f}_{a(b)}(\mathbf{q})\,. \quad (24)$$

The total amplitude for the process $e^+e^- \to J/\psi\,\eta_c$ can be obtained by summing over all the amplitudes of the diagrams shown in Fig. 1.

The unpolarized total cross section (see e.g. Ref. [22]) is obtained by summing over various J/ψ spin-states and averaging over those of the initial state e^+e^-,

$$\sigma = \frac{1}{32\pi}\,\frac{\sqrt{s - 16m_c^2}}{s^{3/2}}\int\frac{1}{4}\sum_{\rm spins}|\mathcal{M}_{\rm total}|^2\,d\cos\theta\,, \quad (25)$$

where the masses of the electron and positron are ignored in the calculation. The explicit expression for the total amplitude $|\mathcal{M}_{\rm total}|^2$, which is the sum of all diagrams shown in Fig. 1, is written as

$$\frac{1}{4}\sum_{\rm spins}|\mathcal{M}_{\rm total}|^2 = \frac{2^{30}\pi^4}{3^4}\,\frac{\alpha_{\rm em}^2\alpha_s^2}{s^5}\,m_c^2(-32m_c^4 + t^2 + u^2) \times \psi_a^2\psi_b^2\,, \quad (26)$$

where $t = (p_1 - P_a)^2$ and $u = (p_1 - P_b)^2$ are the Mandelstam's variables.

Numerical results.

The parameters in the calculation will be taken to be [14, 18]: $a = 2.7183$, $\beta = 0.06$ GeV, $\alpha_{\rm em} \approx 1/137$, $\alpha_s = 0.26$, λ= 0.2 GeV^2, $\Lambda_{\rm QCD} = 0.26$ GeV, $m_c = 1.7753$ GeV. In the interaction kernel $V_0 = 0.415$ GeV for J/ψ and $V_0 = 0.525$ GeV for η_c. With these parameters, we can solve the BS equations numerically and the wave functions $\widetilde{f}_a$ and $\widetilde{f}_b$ are plotted in Fig.2.

Then we have

$$\psi_a = 0.1020\ \text{GeV}\,, \qquad \psi_b = 0.1037\ \text{GeV}. \quad (27)$$

Consequently, the total cross section is obtained as

$$\sigma(e^+e^- \to J/\psi\,\eta_c) = 22.3\ \text{fb} \quad (28)$$

If we vary m_c and α_s within 10% we can get an errors $\pm 1.6 fb$.

The above analysis for the exclusive process $e^+e^- \to J/\psi\,\eta_c$ can be applied, with only a little modifications, to the exclusive process $e^+e^- \to \Upsilon(1S)\,\eta_b$. $\eta_b(9434)$ is the lowest-lying pseudo-scalar $b\bar b$ state (for discussions about

[2] The energy scale μ is of the same order of m_Q and then, in the heavy quark limit, $\alpha_s(\mu) \sim 1/\log(m_Q/\Lambda_{\rm QCD}) \to 0$.

[3] We have rotated the phase of the wave function to make f_a and f_b be real.

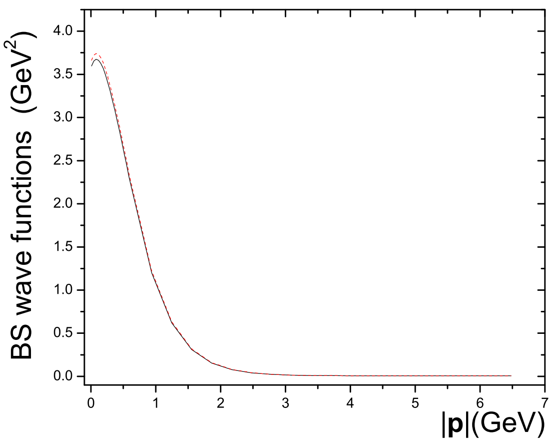

FIG. 2: BS wave functions for J/ψ and η_c in the heavy quark limit, $\widetilde{f}_a$ (solid line) and $\widetilde{f}_b$ (dashed line), respectively.

$\eta_b(9434)$, see e.g. Ref. [23]). The mass of the b quark is $m_b = 5.224$ GeV [18]. In the interaction kernel $V_0 = 0.62$ GeV for $\Upsilon(1S)$ and $V_0 = 0.64$ GeV for η_b. With these parameters, we can solve numerically the wave functions $\widetilde{f}_a$ and $\widetilde{f}_b$, which lead to

$$\psi_a = 0.1123 \text{ GeV}\,, \qquad \psi_b = 0.1124 \text{ GeV}. \tag{29}$$

Then the cross section is predicted to be

$$\sigma(e^+e^- \to \Upsilon(1S)\,\eta_b) = (0.16\,\text{-}\,0.06) \text{ fb} \tag{30}$$

for the range of the total energy $\sqrt{s} = (25\,\text{-}\,30)$ GeV.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we study the exclusive processes of e^+e^- annihilating into two quarkonia in terms of the BS formalism. We find that, in the heavy quark limit, the cross section is $\sigma[e^+e^- \to J/\psi\,\eta_c] = 22.3$ fb, which is compatible with the Babar's data, $\sigma[e^+e^- \to J/\psi\,\eta_c] \geq 17.6 \pm 2.8 \pm 2.1$ fb [2], and the Belle's data, $\sigma[e^+e^- \to J/\psi\,\eta_c] \geq 25.6 \pm 2.8 \pm 3.4$ fb [1]. Because the BS formalism is established based on the relativistic quantum field theory, one has a strong reason to believe that some (perhaps not all) relativistic effects are automatically included in the calculations. The missing part may come from the instantaneous approximation which is necessary for solving the BS equation. Since the approximation is proved to be reasonable in theoretical calculations for other similar processes, we may be convinced that the missing part is not significant. Thus we expect that the non-leading-order contributions, from extra $1/m_Q$ which is indeed a relativistic effect, and α_s corrections, should be small. Our result is different from those given in Refs. [6, 7, 11, 12] in the NRQCD framework, where the leading-order contribution is too small to be comparable with the experimental data. In order to reduce the discrepancy between the experimental data and the theoretical predictions based on NRQCD, large non-leading-order contributions (including α_s or/and relativistic correction(s)) are required in the NRQCD framework and the value of the total non-leading-order correction is nearly twice of the leading-order one.

We also calculate the cross section for the exclusive process with two bottomonia as the final state, $\sigma[e^+e^- \to \Upsilon(1S)\eta_b] = (0.16\,\text{-}\,0.06)$ fb corresponding to the range of the total energy $\sqrt{s} = (25\,\text{-}\,30)$ GeV. Since $m_b \gg m_c$, the non-leading-order contributions, including effects of higher orders in $1/m_b$ and $\alpha_s(2m_b)$ expansions, should be much smaller than those in the charmonium case. Therefore, we expect that the calculation in the bottomonium case is much more precise than that in the charmonium case. Even though such processes cannot be measured at present due to the constraint of the available energy range at the B-factories, the planned ILC will be a powerful facility to testify this result.

Due to the obvious advantage of the BS formalism for dealing with the processes where heavy flavors are involved, we may hope that the obtained results are close to reality.

It will be interesting to study the non-leading-order contributions in the formalism used in this paper and check whether our expectation about the non-leading-order contributions is right. This task is beyond the scope of this paper and will be discussed elsewhere.

Acknowledgments

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